Let D_N denote the Dirichlet kernel $D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta}$.

Show that $\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\,d\theta \geq c\log N$ for some $c > 0$.

Remark: This tells us the Dirichlet kernel is not a good kernel.

Proof: $D_N(\theta) = \sum_{n=-N}^{N} e^{in\theta}$

$$= \sum_{n=0}^{N} e^{in\theta} + \sum_{n=1}^{N} e^{-in\theta}$$

$$= \frac{1-e^{i(N+1)\theta}}{1-e^{i\theta}} + \frac{e^{-i\theta}(1-e^{-iN\theta})}{1-e^{-i\theta}}$$

$$= \frac{e^{i(N+\frac{1}{2})\theta} - e^{-i\frac{1}{2}\theta}}{e^{i\frac{1}{2}\theta} - e^{-i\frac{1}{2}\theta}} + \frac{e^{-i\frac{1}{2}\theta} - e^{-i(N+\frac{1}{2})\theta}}{e^{i\frac{1}{2}\theta} - e^{-i\frac{1}{2}\theta}}$$

$$= \frac{e^{i(N+\frac{1}{2})\theta} - e^{-i(N+\frac{1}{2})\theta}}{e^{i\frac{1}{2}\theta} - e^{-i\frac{1}{2}\theta}}$$

$$= \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$$

Since $|\sin x| \le |x|$, $|D_N(\theta)| \geq \dfrac{2|\sin(N+\frac{1}{2})\theta|}{|\theta|}$.

Then
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\,d\theta \geq \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{|\sin(N+\frac{1}{2})\theta|}{|\theta|}\,d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{|\sin(N+\frac{1}{2})\theta|}{\theta}\,d\theta \qquad \text{even}$$

$$= \frac{2}{\pi}\int_0^{(N+\frac{1}{2})\pi} \frac{|\sin x|}{x/(N+\frac{1}{2})}\,\frac{dx}{N+\frac{1}{2}} \qquad x=(N+\tfrac{1}{2})\theta$$

$$= \frac{2}{\pi}\int_0^{(N+\frac{1}{2})\pi} \frac{|\sin x|}{x}\,dx$$

$$\geq \frac{2}{\pi}\sum_{n=1}^{N}\int_{(n-1)\pi}^{n\pi} \frac{|\sin x|}{x}\,dx$$

$$= \frac{2}{\pi}\sum_{n=1}^{N}\int_0^{\pi} \frac{|\sin(y+(n-1)\pi)|}{y+(n-1)\pi}\,dy \qquad y=x-(n-1)\pi$$

$$= \frac{2}{\pi}\sum_{n=1}^{N}\int_0^{\pi} \frac{\sin y}{y+(n-1)\pi}\,dy$$

$$\geq \frac{2}{\pi}\sum_{n=1}^{N}\frac{1}{n\pi}\int_0^{\pi}\sin y\,dy$$

$$= \frac{4}{\pi^2}\sum_{n=1}^{N}\frac{1}{n}$$

$$\geq \frac{4}{\pi^2}\sum_{n=1}^{N}\int_n^{n+1}\frac{1}{z}\,dz$$

$$= \frac{4}{\pi^2}\int_1^{N+1} \frac{1}{z}\,dz$$

$$= \frac{4}{\pi^2}\log(N+1)$$

$$\geq \frac{4}{\pi^2}\log N$$

$\square$

Recall in HW2, we are asked to show

$$f \in C^k \implies |\hat{f}(n)| = O\left(\frac{1}{|n|^k}\right)$$

Quick review of the proof:

$$|\hat{f}(n)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta\right|$$

$$= \left|\frac{1}{in}\frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}\,d\theta\right| \quad \text{Integral by part}$$

$$\vdots$$

$$= \left|\frac{1}{(in)^k}\frac{1}{2\pi}\int_{-\pi}^{\pi} f^{(k)}(\theta)e^{-in\theta}\,d\theta\right|$$

$$\leq \frac{1}{2\pi|n|^k}\int_{-\pi}^{\pi}|f^{(k)}(\theta)|\,d\theta$$

$$\leq \frac{M}{|n|^k}$$

$f^{(k)}$ is continuous on $[-\pi,\pi]$, thus bounded.

Definition

f is α-Hölder if $\exists$ $C>0$ such that for any x, y,

$|f(x)-f(y)| \leq C|x-y|^{\alpha}$.

Show that

$$f \ \alpha\text{-Hölder} \Rightarrow |\hat{f}(n)| = O\left(\frac{1}{|n|^{\alpha}}\right)$$

Proof: Note that

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$= \frac{1}{2\pi}\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(y+\tfrac{\pi}{n})e^{-in(y+\frac{\pi}{n})}dy \qquad x = y+\frac{\pi}{n}$$

$$= -\frac{1}{2\pi}\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(y+\tfrac{\pi}{n})e^{-iny}dy \qquad e^{-i\pi} = -1$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y+\tfrac{\pi}{n})e^{-iny}dy \qquad 2\pi\text{-periodic}$$

Thus $\hat{f}(n) = \frac{1}{2}\hat{f}(n) + \frac{1}{2}\hat{f}(n)$

$$= \frac{1}{4\pi}\left[\int_{-\pi}^{\pi} f(x)e^{-inx}dx - \int_{-\pi}^{\pi} f(x+\tfrac{\pi}{n})e^{-inx}dx\right]$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left[f(x) - f\left(x + \frac{\pi}{n}\right) \right] e^{-inx} \, dx$$

Therefore, $\left| \hat{f}(n) \right| \leq \frac{1}{4\pi} \int_{-\pi}^{\pi} \left| f(x) - f\left(x + \frac{\pi}{n}\right) \right| dx$

$$\leq \frac{1}{4\pi} \int_{-\pi}^{\pi} C \left(\frac{\pi}{|n|} \right)^{\alpha} dx$$

$$= \frac{C\pi^{\alpha}}{2|n|^{\alpha}}$$

$\square$